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New integrable systems related to the relativistic Toda lattice

Yuri B Suris†

Centre for Complex Systems and Visualization, University of Bremen, Universitätsallee 29, 28359 Bremen, Germany

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Abstract. The bi–Hamiltonian structure of the relativistic Toda lattice is exploited to introduce some new integrable lattice systems. Their integrable discretizations are obtained by means of the general procedure proposed recently by the author. Bäcklund transformations between the new systems and the relativistic Toda lattice (in both the continuous and discrete time formulations) are established.

1. Introduction

This paper is devoted to integrable equations of classical mechanics. More precisely, we shall deal here with equations of motion in the Newtonian form.

We introduce two new integrable continuous time lattice systems, and present several novel integrable discrete time systems.

The first continuous time system is

$$\ddot{x}_k = \dot{x}_{k+1}\exp(x_{k+1}-x_k) - \exp(2(x_{k+1}-x_k)) - \dot{x}_{k-1}\exp(x_k-x_{k-1}) + \exp(2(x_k-x_{k-1})). \tag{1.1}$$

The second is

$$\ddot{x}_k = -\dot{x}_k^2\big(\dot{x}_{k+1}\exp(x_{k+1}-x_k) - \dot{x}_{k-1}\exp(x_k-x_{k-1})\big). \tag{1.2}$$

To the author's knowledge, these systems have not appeared in the literature, despite their beauty and possible physical applications. However, they are closely related to another well known integrable lattice, namely the relativistic Toda lattice:

$$\ddot{x}_k = \dot{x}_{k+1}\dot{x}_k\frac{g^2\exp(x_{k+1}-x_k)}{1+g^2\exp(x_{k+1}-x_k)} - \dot{x}_k\dot{x}_{k-1}\frac{g^2\exp(x_k-x_{k-1})}{1+g^2\exp(x_k-x_{k-1})}. \tag{1.3}$$

More precisely, we shall find a kind of Bäcklund transformation connecting (1.1) and (1.3), and another one connecting (1.2) and (1.3).

We now write down integrable discretizations we propose for the lattices (1.1), (1.2). In the difference equations below, $x_k = x_k(t)$ are supposed to be functions of the discrete

† E-mail address: suris@cevis.uni-bremen.de

time $t \in h\mathbb{Z}$, and $\widetilde{x}_k = x_k(t+h)$, $\underset{\sim}{x}_k = x_k(t-h)$. Discretization of the first lattice (1.1) yields

$$\exp(\widetilde{x}_k - x_k) - \exp(x_k - \underset{\sim}{x}_k) = -\frac{1}{1 - h\exp(\underset{\sim}{x}_{k+1} - x_k)} + h\exp(x_{k+1} - x_k) + \frac{1}{1 - h\exp(x_k - \widetilde{x}_{k-1})} - h\exp(x_k - x_{k-1}). \tag{1.4}$$

Discretization of the second lattice (1.2) yields

$$\frac{h}{\exp(\widetilde{x}_k - x_k) - 1} - \frac{h}{\exp(x_k - \underset{\sim}{x}_k) - 1} = \exp(x_{k+1} - x_k) - \exp(\underset{\sim}{x}_{k+1} - x_k) - \exp(x_k - x_{k-1}) + \exp(x_k - \widetilde{x}_{k-1}). \tag{1.5}$$

The same Bäcklund transformations as for the continuous time systems relate these systems of difference equations to the discrete time relativistic Toda lattice [1]:

$$\frac{\exp(\widetilde{x}_k - x_k) - 1}{\exp(x_k - \underset{\sim}{x}_k) - 1} = \frac{\big(1 + g^2\exp(x_{k+1} - x_k)\big)}{\big(1 + g^2\exp(\underset{\sim}{x}_{k+1} - x_k)\big)} \frac{\big(1 + g^2\exp(x_k - \widetilde{x}_{k-1})\big)}{\big(1 + g^2\exp(x_k - x_{k-1})\big)}. \tag{1.6}$$

A modification of the construction leading to the above discrete time systems allows us to derive several further nice discretizations. For example, for the lattice (1.1) we have

$$\exp(\widetilde{x}_k - 2x_k + \underset{\sim}{x}_k) = \frac{\big(1 + h\exp(x_{k+1} - x_k)\big)\big(1 - h\exp(\underset{\sim}{x}_{k+1} - x_k)\big)}{\big(1 + h\exp(x_k - x_{k-1})\big)\big(1 - h\exp(x_k - \widetilde{x}_{k-1})\big)} \tag{1.7}$$

and for the relativistic Toda lattice (1.3) we have

$$\frac{\exp(-\widetilde{x}_k + x_k) - 1}{\exp(-x_k + \underset{\sim}{x}_k) - 1} = \frac{\big(1 + g^2\exp(x_{k+1} - x_k)\big)}{\big(1 + g^2\exp(\underset{\sim}{x}_{k+1} - x_k)\big)} \frac{\big(1 + g^2\exp(x_k - \widetilde{x}_{k-1})\big)}{\big(1 + g^2\exp(x_k - x_{k-1})\big)}. \tag{1.8}$$

(The last system resembles the previous discretization of the relativistic Toda lattice (1.6) very closely; however, the relation between them is far from trivial).

All the above systems (continuous and discrete time) may be considered either on an infinite lattice ($k \in \mathbb{Z}$), or on a finite one ($1 \leqslant k \leqslant N$). In the latter case one of two types of boundary conditions may be imposed: open-end ($x_0 = \infty$, $x_{N+1} = -\infty$) or periodic ($x_0 \equiv x_N$, $x_{N+1} \equiv x_1$). We shall be concerned only with finite lattices here, consideration of the infinite ones being to a large extent similar.

One remark: the 'list' of references at the end of this paper might look strange; in fact most of the references cited in [1] are relevant, and we omit them here solely to save space. The interested reader is advised to consult these references.

2. The simplest flows of the relativistic Toda hierarchy and their bi-Hamiltonian structure

In this section we consider the two simplest flows of the relativistic Toda hierarchy. All the results here and in section 3 are not new, but are collected in the form convenient for our present purposes. For the relevant references see [1].

The first flow of the relativistic Toda hierarchy is

$$\dot{d}_k = d_k(c_k - c_{k-1}) \qquad \dot{c}_k = c_k(d_{k+1} + c_{k+1} - d_k - c_{k-1}). \tag{2.1}$$

The second flow of the relativistic Toda hierarchy is

$$\dot{d}_k = d_k \left(\frac{c_k}{d_k d_{k+1}} - \frac{c_{k-1}}{d_{k-1} d_k} \right) \qquad \dot{c}_k = c_k \left(\frac{1}{d_k} - \frac{1}{d_{k+1}} \right). \tag{2.2}$$

They may be considered either under open-end boundary conditions ($d_{N+1} = c_0 = c_N = 0$), or under periodic ones (all the subscripts are taken (mod N), so that $d_{N+1} \equiv d_1$, $c_0 \equiv c_N$, $c_{N+1} \equiv c_1$).

We now discuss a Hamiltonian structure for both flows (2.1), (2.2). It is easy to see that they are Hamiltonian with respect to two different compatible Poisson brackets. The first bracket is linear, namely

$$\{c_k, d_{k+1}\}_1 = -c_k \qquad \{c_k, d_k\}_1 = c_k \qquad \{d_k, d_{k+1}\}_1 = c_k \tag{2.3}$$

(only the non-vanishing brackets are written down), and the Hamiltonian functions generating the flows (2.1), (2.2) in this bracket are equal to

$$H_+^{(1)} = \frac{1}{2} \sum_{k=1}^{N} (d_k + c_{k-1})^2 + \sum_{k=1}^{N} (d_k + c_{k-1}) c_k \qquad H_-^{(1)} = -\sum_{k=1}^{N} \log(d_k). \tag{2.4}$$

The second Poisson bracket is quadratic, namely

$$\{c_k, c_{k+1}\}_2 = -c_k c_{k+1} \qquad \{c_k, d_{k+1}\}_2 = -c_k d_{k+1} \qquad \{c_k, d_k\}_2 = c_k d_k \tag{2.5}$$

the corresponding Hamiltonian functions being

$$H_+^{(2)} = \sum_{k=1}^{N} (d_k + c_k) \qquad H_-^{(2)} = \sum_{k=1}^{N} \frac{d_k + c_k}{d_k d_{k+1}}. \tag{2.6}$$

We now turn to the integrable discretizations of the flows (2.1), (2.2) derived in [1].

An integrable discretization of the flow (2.1) is given by the difference equations

$$\widetilde{d}_k = d_k \frac{\mathfrak{a}_{k+1} - h d_{k+1}}{\mathfrak{a}_k - h d_k} \qquad \widetilde{c}_k = c_k \frac{\mathfrak{a}_{k+1} + h c_{k+1}}{\mathfrak{a}_k + h c_k} \tag{2.7}$$

where $\mathfrak{a}_k = \mathfrak{a}_k(c, d)$ is defined as a unique set of functions satisfying the recurrence relation

$$\mathfrak{a}_k = 1 + h d_k + \frac{h c_{k-1}}{\mathfrak{a}_{k-1}} \tag{2.8}$$

together with an asymptotic relation

$$\mathfrak{a}_k = 1 + h(d_k + c_{k-1}) + \mathrm{O}(h^2). \tag{2.9}$$

In the open-end case, due to $c_0 = 0$, from (2.8) we obtain the following finite continued fractions expressions for $\mathfrak{a}_k$:

$$\mathfrak{a}_1 = 1 + h d_1 \qquad \mathfrak{a}_2 = 1 + h d_2 + \frac{h c_1}{1 + h d_1}$$

$$\cdots \qquad \mathfrak{a}_N = 1 + h d_N + \cfrac{h c_{N-1}}{1 + h d_{N-1} + \cfrac{h c_{N-2}}{1 + h d_{N-2} + \ddots + \cfrac{h c_1}{1 + h d_1}}}.$$

In the periodic case equations (2.8), (2.9) uniquely define the $\mathfrak{a}_k$ as N-periodic infinite continued fractions. It can be proved that for h sufficiently small these continued fractions converge and their values satisfy (2.9).

An integrable discretization of the flow (2.2) is given by the difference equations

$$\widetilde{d}_k = d_{k+1}\frac{d_k - h\mathfrak{d}_{k-1}}{d_{k+1} - h\mathfrak{d}_k} \qquad \widetilde{c}_k = c_{k+1}\frac{c_k + h\mathfrak{d}_k}{c_{k+1} + h\mathfrak{d}_{k+1}} \tag{2.10}$$

where $\mathfrak{d}_k = \mathfrak{d}_k(c, d)$ is defined as a unique set of functions satisfying the recurrence relation

$$\frac{c_k}{\mathfrak{d}_k} = d_k - h - h\mathfrak{d}_{k-1} \tag{2.11}$$

together with an asymptotic relation

$$\mathfrak{d}_k = \frac{c_k}{d_k} + \mathrm{O}(h). \tag{2.12}$$

In the open-end case we obtain from (2.11) the following finite continued fractions expressions for $\mathfrak{d}_k$:

$$\mathfrak{d}_1 = \frac{c_1}{d_1 - h} \qquad \mathfrak{d}_2 = \cfrac{c_2}{d_2 - h - \cfrac{hc_1}{d_1 - h}}$$

$$\cdots \qquad \mathfrak{d}_{N-1} = \cfrac{c_{N-1}}{d_{N-1} - h - \cfrac{hc_{N-2}}{d_{N-2} - h - \ddots - \cfrac{hc_1}{d_1 - h}}}.$$

In the periodic case (2.11), (2.12) uniquely define the $\mathfrak{d}_k$ as N-periodic infinite continued fractions. It can be proved that for h sufficiently small these continued fractions converge and their values satisfy (2.12).

It can be proved [1] that the maps (2.7), (2.10) are Poisson with respect to both brackets (2.3), (2.5).

3. Lax representations

Recall [1] that both the continuous time systems (2.1), (2.2) and discrete time systems (2.7), (2.10) admit Lax representations, the Lax matrices being the same for the both cases.

The following statement holds. Introduce two N by N matrices depending on the phase space coordinates c_k, d_k and (in the periodic case) on the additional parameter λ:

$$L(c, d, \lambda) = \sum_{k=1}^{N} d_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \tag{3.1}$$

$$U(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}. \tag{3.2}$$

Here E_{jk} stands for the matrix whose only non-zero entry at the intersection of the jth row and the kth column is equal to 1. In the periodic case we have $E_{N+1,N} = E_{1,N}$, $E_{N,N+1} = E_{N,1}$; in the open-end case we set $\lambda = 1$, and $E_{N+1,N} = E_{N,N+1} = 0$. Consider also the following two matrices:

$$T_+(c, d, \lambda) = L(c, d, \lambda)U^{-1}(c, d, \lambda) \qquad T_-(c, d, \lambda) = U^{-1}(c, d, \lambda)L(c, d, \lambda). \tag{3.3}$$

Proposition 1. The flow (2.1) is equivalent to the following matrix differential equations:

$$\dot{L} = LB - AL \qquad \dot{U} = UB - AU \tag{3.4}$$

which also imply that

$$\dot{T}_+ = [T_+, A] \qquad \dot{T}_- = [T_-, B] \tag{3.5}$$

where

$$A(c, d, \lambda) = \sum_{k=1}^{N} (d_k + c_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \tag{3.6}$$

$$B(c, d, \lambda) = \sum_{k=1}^{N} (d_k + c_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}. \tag{3.7}$$

Proposition 2. The map (2.7) is equivalent to the following matrix difference equations:

$$\widetilde{L} = \mathbf{A}^{-1} L \mathbf{B} \qquad \widetilde{U} = \mathbf{A}^{-1} U \mathbf{B} \tag{3.8}$$

which also imply that

$$\widetilde{T}_+ = \mathbf{A}^{-1} T_+ \mathbf{A} \qquad \widetilde{T}_- = \mathbf{B}^{-1} T_- \mathbf{B} \tag{3.9}$$

where

$$\mathbf{A}(c, d, \lambda) = \sum_{k=1}^{N} \mathfrak{a}_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \tag{3.10}$$

$$\mathbf{B}(c, d, \lambda) = \sum_{k=1}^{N} \mathfrak{b}_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k} \tag{3.11}$$

and the quantities $\mathfrak{b}_k$ are defined by

$$\mathfrak{b}_k = \mathfrak{a}_k \frac{\mathfrak{a}_{k+1} - h d_{k+1}}{\mathfrak{a}_k - h d_k} = \mathfrak{a}_{k-1} \frac{\mathfrak{a}_k + h c_k}{\mathfrak{a}_{k-1} + h c_{k-1}}. \tag{3.12}$$

Note that the compatibility of the two expressions for $\mathfrak{b}_k$ in (3.12) is an immediate consequence of (2.8), and that from (3.12), (2.9) it follows that

$$\mathfrak{b}_k = 1 + h(d_k + c_k) + \mathrm{O}(h^2). \tag{3.13}$$

Proposition 3. The flow (2.2) is equivalent to the following matrix differential equations:

$$\dot{L} = LD - CL \qquad \dot{U} = UD - CU \tag{3.14}$$

which also imply that

$$\dot{T}_+ = [T_+, C] \qquad \dot{T}_- = [T_-, D] \tag{3.15}$$

where

$$C(c, d) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_{k+1}} E_{k,k+1} \tag{3.16}$$

$$D(c, d) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_k} E_{k,k+1}. \tag{3.17}$$

Proposition 4. The map (2.10) is equivalent to the following matrix difference equations:

$$\widetilde{L} = \mathbf{C}L\mathbf{D}^{-1} \qquad \widetilde{U} = \mathbf{C}U\mathbf{D}^{-1} \tag{3.18}$$

which also imply that

$$\widetilde{T}_+ = \mathbf{C}T_+\mathbf{C}^{-1} \qquad \widetilde{T}_- = \mathbf{D}T_-\mathbf{D}^{-1} \tag{3.19}$$

where

$$\mathbf{C}(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1}\sum_{k=1}^{N} \mathfrak{c}_k E_{k,k+1} \tag{3.20}$$

$$\mathbf{D}(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1}\sum_{k=1}^{N} \mathfrak{d}_k E_{k,k+1} \tag{3.21}$$

and the quantities $\mathfrak{c}_k$ are defined by

$$\mathfrak{c}_k = \mathfrak{d}_k \frac{d_k - h\mathfrak{d}_{k-1}}{d_{k+1} - h\mathfrak{d}_k} = \mathfrak{d}_{k+1}\frac{c_k + h\mathfrak{d}_k}{c_{k+1} + h\mathfrak{d}_{k+1}}. \tag{3.22}$$

The compatibility of the two expressions for $\mathfrak{c}_k$ in (3.22) is an immediate consequence of (2.11), and from (3.22), (2.12) it follows that

$$\mathfrak{c}_k = \frac{c_k}{d_{k+1}} + \mathrm{O}(h). \tag{3.23}$$

The spectral invariants of the matrices $T_\pm(c, d, \lambda)$ serve as integrals of motion for the flows (2.1), (2.2), as well as for the maps (2.7), (2.10). In particular, it is easy to see that the Hamiltonian functions (2.4), (2.6) are spectral invariants of the Lax matrices:

$$H_+^{(1)} = \tfrac{1}{2}\,\mathrm{tr}(T_\pm^2) \qquad H_-^{(1)} = -\,\mathrm{tr}\log(T_\pm)$$

$$H_+^{(2)} = \mathrm{tr}(T_\pm) \qquad H_-^{(2)} = \mathrm{tr}(T_\pm^{-1}).$$

Moreover, it can be proved [1] that the maps (2.7), (2.10) admit interpolation in both Poisson brackets (2.3), (2.5) by Hamiltonian flows, the Hamiltonian fuctions being certain spectral invariants of the matrices $T_\pm$.

4. Parametrization of the linear bracket by canonically conjugate variables

In what follows we shall consider different Poisson maps from the standard symplectic space $\mathbb{R}^{2N}(x, p)$ into the Poisson space $\mathbb{R}^{2N}(c, d)$, the latter being equipped with different Poisson brackets, the former always being equipped with the canonical brackets

$$\{x_k, x_j\} = \{p_k, p_j\} = 0 \qquad \{p_k, x_j\} = \delta_{kj}. \tag{4.1}$$

We shall call such maps *parametrizations* of the corresponding Poisson bracket on $\mathbb{R}^{2N}(c, d)$ through canonically conjugate variables (x, p).

For example, the linear Poisson bracket (2.3) may be parametrized by the canonically conjugate variables (x, p) according to the formulae

$$d_k = p_k - \exp(x_k - x_{k-1}) \qquad c_k = \exp(x_{k+1} - x_k). \tag{4.2}$$

Let us see how the equations of motion look in this parametrization. We start with (2.1), (2.7).

Obviously, the function $H_+^{(1)}$ takes the form

$$H_+^{(1)} = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} p_k \exp(x_{k+1} - x_k). \tag{4.3}$$

Correspondingly, the flow (2.1) takes the form of canonical equations of motion

$$\dot{x}_k = \partial H_+^{(1)}/\partial p_k = p_k + \exp(x_{k+1} - x_k)$$

$$\dot{p}_k = -\partial H_+^{(1)}/\partial x_k = p_k \exp(x_{k+1} - x_k) - p_{k-1} \exp(x_k - x_{k-1}).$$

As an immediate consequence of these equations one obtains the Newtonian equations of motion (1.1). A standard procedure allows one to find a Lagrangian formulation of these equations. Indeed, one has to express

$$\mathcal{L} = \sum_{k=1}^{N} \dot{x}_k p_k - H \tag{4.4}$$

in terms of $(x_k, \dot{x}_k)$, which in the present case leads to

$$\mathcal{L}_+^{(1)}(x, \dot{x}) = \frac{1}{2}\sum_{k=1}^{N} \big(\dot{x}_k - \exp(x_{k+1} - x_k)\big)^2. \tag{4.5}$$

Note that the results of section 3 provide us with a Lax representation of our new lattice (1.1): in the formulae of proposition 1 one need only set

$$c_k = \exp(x_{k+1} - x_k) \qquad d_k = \dot{x}_k - \exp(x_k - x_{k-1}) - \exp(x_{k+1} - x_k). \tag{4.6}$$

We now turn to the less straightforward case of discrete equations of motion.

Theorem 1. In the parametrization (4.2) the equations of motion (2.7) may be presented in the form of the following two equations:

$$hp_k = \exp(\widetilde{x}_k - x_k) - \frac{1}{1 - h\exp(x_k - \widetilde{x}_{k-1})} + h\exp(x_k - x_{k-1}) - h\exp(x_{k+1} - x_k) \tag{4.7}$$

$$h\widetilde{p}_k = \exp(\widetilde{x}_k - x_k) - \frac{1}{1 - h\exp(x_{k+1} - \widetilde{x}_k)} \tag{4.8}$$

which also imply the Newtonian equations of motion (1.4).

Proof. The second equation of motion in (2.7), together with the parametrization $c_k = \exp(x_{k+1} - x_k)$, implies that the following quantity is constant, i.e. it does not depend on k:

$$\exp(-\widetilde{x}_k + x_k)(\mathfrak{a}_k + hc_k) = \text{constant}.$$

Choosing this constant to be equal to 1, we obtain

$$\mathfrak{a}_k + hc_k = \exp(\widetilde{x}_k - x_k) \tag{4.9}$$

hence

$$\mathfrak{a}_k = \exp(\widetilde{x}_k - x_k) - h\exp(x_{k+1} - x_k) = \exp(\widetilde{x}_k - x_k)\big(1 - h\exp(x_{k+1} - \widetilde{x}_k)\big). \tag{4.10}$$

Substituting the last two formulae in (2.8), we obtain

$$\mathfrak{a}_k - hd_k = 1 + \frac{hc_{k-1}}{\mathfrak{a}_{k-1}} = \frac{1}{1 - h\exp(x_k - \widetilde{x}_{k-1})} \tag{4.11}$$

or

$$hd_k = \exp(\widetilde{x}_k - x_k)\big(1 - h\exp(x_{k+1} - \widetilde{x}_k)\big) - \frac{1}{1 - h\exp(x_k - \widetilde{x}_{k-1})}. \tag{4.12}$$

Now the first equation of motion in (2.7) may be rewritten with the help of (4.11) as

$$\widetilde{d}_k = d_k \frac{1 - h\exp(x_k - \widetilde{x}_{k-1})}{1 - h\exp(x_{k+1} - \widetilde{x}_k)}$$

which, together with (4.24), implies

$$h\widetilde{d}_k = \exp(\widetilde{x}_k - x_k)\big(1 - h\exp(x_k - \widetilde{x}_{k-1})\big) - \frac{1}{1 - h\exp(x_{k+1} - \widetilde{x}_k)}. \tag{4.13}$$

Under the parametrization $d_k = p_k - \exp(x_k - x_{k-1})$ equations (4.12), (4.13) are equivalent to (4.7), (4.8). □

The Lax representations for the system (1.4) is given by proposition 2, where the expressions for the coefficients c_k, d_k, $\mathfrak{a}_k$, $\mathfrak{b}_k$ in terms of the variables x_k and their discrete time updates $\widetilde{x}_k$ are given by $c_k = \exp(x_{k+1} - x_k)$, (4.12), (4.13), (4.10), and

$$\mathfrak{b}_k = \exp(\widetilde{x}_k - x_k)\big(1 - h\exp(x_k - \widetilde{x}_{k-1})\big).$$

(the last formula following from (3.12), (4.10), and (4.11)).

Note also that equations (4.7), (4.8) not only immediately imply (1.4) from the introduction, but, moreover, allow one to find a Lagrangian interpretation of this equation. Indeed, the general theory says that if the equations of motion are represented in the Lagrange form

$$\partial\big(\Lambda(\widetilde{x}, x) + \Lambda(x, \underset{\sim}{x})\big)/\partial x_k = 0 \tag{4.14}$$

then the momenta p_k canonically conjugate to x_k are given by

$$p_k = -\partial\Lambda(\widetilde{x}, x)/\partial x_k \tag{4.15}$$

so that

$$\widetilde{p}_k = \partial\Lambda(\widetilde{x}, x)/\partial\widetilde{x}_k. \tag{4.16}$$

Identifying equations (4.7) and (4.8) with (4.15) and (4.16), respectively, we see that the Lagrange function for equation (1.4) can be chosen in the form

$$\Lambda_+^{(1)}(\widetilde{x}, x) = \sum_{k=1}^{N} \varphi(\widetilde{x}_k - x_k) - h^{-1}\sum_{k=1}^{N} \log\big(1 - h\exp(x_{k+1} - \widetilde{x}_k)\big) - \sum_{k=1}^{N} \exp(x_{k+1} - x_k) \tag{4.17}$$

where

$$\varphi(\xi) = h^{-1}\big(\exp(\xi) - 1 - \xi\big).$$

Obviously, this function serves as a finite difference approximation to (4.5).

We now turn to the equations of motion (2.2), (2.10), and find out how they look in the parametrization (4.2).

The function $H_-^{(1)}$ takes the form

$$H_-^{(1)} = -\sum_{k=1}^{N} \log\big(p_k - \exp(x_k - x_{k-1})\big). \tag{4.18}$$

Correspondingly, the canonical equations of motion for the flow (2.2) take the form

$$\dot{x}_k = \frac{\partial H_-^{(1)}}{\partial p_k} = -\frac{1}{p_k - \exp(x_k - x_{k-1})}$$

$$\dot{p}_k = \frac{-\partial H_-^{(1)}}{\partial x_k} = \frac{\exp(x_{k+1} - x_k)}{p_{k+1} - \exp(x_{k+1} - x_k)} - \frac{\exp(x_k - x_{k-1})}{p_k - \exp(x_k - x_{k-1})}.$$

As a consequence of these equations, one obtains

$$p_k = -\frac{1}{\dot{x}_k} + \exp(x_k - x_{k-1}) \qquad \dot{p}_k = -\dot{x}_{k+1}\exp(x_{k+1} - x_k) + \dot{x}_k \exp(x_k - x_{k-1})$$

and the Newtonian equations of motion (1.2) follow. A standard procedure, equation (4.4), leads to a Lagrangian formulation of these equations. One has

$$\mathcal{L}_-^{(1)}(x, \dot{x}) = -\sum_{k=1}^{N} \log(\dot{x}_k) + \sum_{k=1}^{N} \dot{x}_k \exp(x_k - x_{k-1}). \tag{4.19}$$

In order to get a Lax representation of the lattice (1.2) one need only set

$$c_k = \exp(x_{k+1} - x_k) \qquad d_k = -\frac{1}{\dot{x}_k} \tag{4.20}$$

in the formulae of proposition 3.

Turning to the discrete equations of motion (2.10), we obtain:

Theorem 2. In the parametrization (4.2) the equations of motion (2.10) may be presented in the form of the following two equations:

$$p_k = -\frac{h}{\exp(\widetilde{x}_k - x_k) - 1} + \exp(x_k - \widetilde{x}_{k-1}) \tag{4.21}$$

$$\widetilde{p}_k = -\frac{h}{\exp(\widetilde{x}_k - x_k) - 1} + \exp(x_{k+1} - \widetilde{x}_k) - \exp(\widetilde{x}_{k+1} - \widetilde{x}_k) + \exp(\widetilde{x}_k - \widetilde{x}_{k-1}) \tag{4.22}$$

which also imply the Newtonian equations of motion (1.4).

Proof. The second equation of motion in (2.10), rewritten as

$$\widetilde{c}_k = c_k \frac{1 + h\mathfrak{d}_k/c_k}{1 + h\mathfrak{d}_{k+1}/c_{k+1}}$$

together with the parametrization $c_k = \exp(x_{k+1} - x_k)$, implies that the following quantity is constant, i.e. it does not depend on k:

$$\exp(\widetilde{x}_k - x_k)\left(1 + \frac{h\mathfrak{d}_k}{c_k}\right) = \text{constant}.$$

Choosing this constant to be equal to 1, we obtain

$$\frac{c_k}{\mathfrak{d}_k} + h = \frac{h}{\exp(\widetilde{x}_k - x_k) - 1} \tag{4.23}$$

hence

$$h\mathfrak{d}_k = \exp(x_{k+1} - \widetilde{x}_k) - \exp(x_{k+1} - x_k) = -\exp(x_{k+1} - \widetilde{x}_k)\big(\exp(\widetilde{x}_k - x_k) - 1\big). \tag{4.24}$$

The recurrence relation (2.11) implies

$$d_k - h\mathfrak{d}_{k-1} = \frac{c_k}{\mathfrak{d}_k} + h \tag{4.25}$$

which, together with (4.24), (4.23), implies

$$d_k = -\frac{h}{\exp(\widetilde{x}_k - x_k) - 1} - \exp(x_k - \widetilde{x}_{k-1})\big(\exp(\widetilde{x}_{k-1} - x_{k-1}) - 1\big). \qquad (4.26)$$

Now we can rewrite the first equation of motion in (2.10), taking into account (4.25), (4.23):

$$\widetilde{d}_k = d_{k+1}\,\frac{\exp(\widetilde{x}_{k+1} - x_{k+1}) - 1}{\exp(\widetilde{x}_k - x_k) - 1}.$$

The last equation, together with (4.26), implies

$$\widetilde{d}_k = -\frac{h}{\exp(\widetilde{x}_k - x_k) - 1} - \exp(x_{k+1} - \widetilde{x}_k)\big(\exp(\widetilde{x}_{k+1} - x_{k+1}) - 1\big). \qquad (4.27)$$

Under the parametrization $d_k = p_k - \exp(x_k - x_{k-1})$, equations (4.26), (4.27) are equivalent to (4.21), (4.22). □

The Lax representations for the system (1.5) is given by proposition 4, where the expressions for the coefficients c_k, d_k, $\mathfrak{d}_k$, $\mathfrak{c}_k$ in terms of the variables x_k and their discrete time updates $\widetilde{x}_k$ are given by $c_k = \exp(x_{k+1} - x_k)$, (4.26), (4.27), (4.24), and

$$h\mathfrak{c}_k = -\exp(x_{k+1} - \widetilde{x}_k)\big(\exp(\widetilde{x}_{k+1} - x_{k+1}) - 1\big)$$

(the last formula following from (4.24), (3.22) and (4.23)).

Identifying equations (4.21), (4.22) with (4.15), (4.16), respectively, we obtain the Lagrange function for equation (1.5) in the form

$$\Lambda_-^{(1)}(\widetilde{x}, x) = -h\sum_{k=1}^{N}\psi(\widetilde{x}_k - x_k) + \sum_{k=1}^{N}\big[\exp(\widetilde{x}_k - \widetilde{x}_{k-1}) - \exp(x_k - \widetilde{x}_{k-1})\big]$$

where

$$\psi(\xi) = \int_0^{\xi}\frac{\mathrm{d}\eta}{\exp(\eta) - 1} = \log(\exp(\xi) - 1) - \xi.$$

This function clearly is a finite difference approximation of (4.19).

5. Parametrization of the quadratic bracket by canonically conjugate variables

For completeness we give the results corresponding to another parametrization of the variables c_k, d_k by means of canonically conjugate variables x_k, p_k, namely the parametrization leading to the quadratic bracket (2.5). The relativistic Toda lattice arises in this manner. The corresponding formulae were given in [1], but in an *ad hoc* manner, without derivation. We take the opportunity of filling in this gap here.

The parametrization leading to the quadratic bracket (2.5) reads

$$d_k = \exp(p_k) \qquad c_k = g^2\exp(x_{k+1} - x_k + p_k) \qquad (5.1)$$

where $g^2 \in \mathbb{R}$ is a coupling constant.

In terms of these variables

$$H_+^{(2)} = \sum_{k=1}^{N}\exp(p_k)\big(1 + g^2\exp(x_{k+1} - x_k)\big)$$
$$H_-^{(2)} = \sum_{k=1}^{N}\exp(-p_k)\big(1 + g^2\exp(x_k - x_{k-1})\big). \qquad (5.2)$$

Hence the equations of motion corresponding to (2.1) take the canonical form

$$\dot{x}_k = \partial H_+^{(2)}/\partial p_k = \exp(p_k)\big(1 + g^2 \exp(x_{k+1} - x_k)\big)$$

$$\dot{p}_k = -\partial H_+^{(2)}/\partial x_k = g^2 \exp(x_{k+1} - x_k + p_k) - g^2 \exp(x_k - x_{k-1} + p_{k-1}).$$

This can be put into a Newtonian form (1.3).

A standard procedure, equation (4.4), allows also to find a Lagrangian formulation of these equations, with a Lagrange function

$$\mathcal{L}_+^{(2)}(x, \dot{x}) = \sum_{k=1}^{N} [\dot{x}_k \log(\dot{x}_k) - \dot{x}_k] - \sum_{k=1}^{N} \dot{x}_k \log\big(1 + g^2 \exp(x_{k+1} - x_k)\big). \tag{5.3}$$

The Lax representation for these equations are given by proposition 1 with

$$d_k = \dot{x}_k/\big(1 + g^2 \exp(x_{k+1} - x_k)\big) \qquad c_k = g^2 \exp(x_{k+1} - x_k) d_k. \tag{5.4}$$

Analogously, the canonical equations of motion corresponding to (2.2) are

$$\dot{x}_k = \partial H_-^{(2)}/\partial p_k = -\exp(-p_k)\big(1 + g^2 \exp(x_k - x_{k-1})\big)$$

$$\dot{p}_k = -\partial H_-^{(2)}/\partial x_k = g^2 \exp(x_{k+1} - x_k - p_{k+1}) - g^2 \exp(x_k - x_{k-1} - p_k).$$

The Newtonian equations following from these are just the same as before (equation (1.3)). They correspond, however, to a different form of Lagrange function:

$$\mathcal{L}_-^{(2)}(x, \dot{x}) = -\sum_{k=1}^{N} [\dot{x}_k \log(-\dot{x}_k) - \dot{x}_k] + \sum_{k=1}^{N} \dot{x}_k \log\big(1 + g^2 \exp(x_k - x_{k-1})\big). \tag{5.5}$$

The Lax representation for these equations is given by proposition 3 with the identifications

$$d_k = -\frac{1 + g^2 \exp(x_k - x_{k-1})}{\dot{x}_k} \qquad c_k = g^2 \exp(x_{k+1} - x_k) d_k. \tag{5.6}$$

We now turn to the discrete time systems (2.7), (2.10).

Theorem 3. In the parametrization (5.1) the map (2.7) takes the form of the following two equations:

$$h \exp(p_k) = \frac{\big(\exp(\widetilde{x}_k - x_k) - 1\big)}{\big(1 + g^2 \exp(x_{k+1} - x_k)\big)} \frac{\big(1 + g^2 \exp(x_k - x_{k-1})\big)}{\big(1 + g^2 \exp(x_k - \widetilde{x}_{k-1})\big)} \tag{5.7}$$

$$h \exp(\widetilde{p}_k) = \frac{\big(\exp(\widetilde{x}_k - x_k) - 1\big)}{\big(1 + g^2 \exp(x_{k+1} - \widetilde{x}_k)\big)}. \tag{5.8}$$

This also implies a Newtonian form (1.6) of the equations of motion.

Proof. From equations (2.7) it follows that

$$\frac{\widetilde{c}_k}{\widetilde{d}_k} = \frac{c_k}{d_k} \frac{\mathfrak{a}_{k+1} + hc_{k+1}}{\mathfrak{a}_{k+1} - hd_{k+1}} \frac{\mathfrak{a}_k - hd_k}{\mathfrak{a}_k + hc_k}.$$

Since $c_k/d_k = g^2 \exp(x_{k+1} - x_k)$, this implies that the following quantity is constant, i.e. it does not depend on k:

$$\exp(\widetilde{x}_k - x_k) \frac{\mathfrak{a}_k - hd_k}{\mathfrak{a}_k + hc_k} = \text{constant}.$$

Setting this constant equal to 1, we obtain

$$\frac{\mathfrak{a}_k + hc_k}{\mathfrak{a}_k - hd_k} = \exp(\widetilde{x}_k - x_k).$$

This implies

$$\mathfrak{a}_k = h\frac{d_k \exp(\widetilde{x}_k - x_k) + c_k}{\exp(\widetilde{x}_k - x_k) - 1}$$
$$= hd_k \frac{\exp(\widetilde{x}_k - x_k)\big(1 + g^2 \exp(x_{k+1} - \widetilde{x}_k)\big)}{\exp(\widetilde{x}_k - x_k) - 1}. \tag{5.9}$$

As a consequence, we obtain

$$\mathfrak{a}_k + hc_k = h\exp(\widetilde{x}_k - x_k)\frac{d_k + c_k}{\exp(\widetilde{x}_k - x_k) - 1}$$
$$= hd_k \frac{\exp(\widetilde{x}_k - x_k)\big(1 + g^2 \exp(x_{k+1} - x_k)\big)}{\exp(\widetilde{x}_k - x_k) - 1}. \tag{5.10}$$

Substituting equations (5.9), (5.10) in the recurrence relation (2.8), we obtain

$$\mathfrak{a}_k - hd_k = 1 + \frac{hc_{k-1}}{\mathfrak{a}_{k-1}} = \frac{1 + g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - \widetilde{x}_{k-1})}. \tag{5.11}$$

Substituting expression (5.9) for $\mathfrak{a}_k$ in the left-hand side of this formula, we arrive at

$$hd_k = \frac{\big(\exp(\widetilde{x}_k - x_k) - 1\big)}{\big(1 + g^2 \exp(x_{k+1} - x_k)\big)}\ \frac{\big(1 + g^2 \exp(x_k - x_{k-1})\big)}{\big(1 + g^2 \exp(x_k - \widetilde{x}_{k-1})\big)}. \tag{5.12}$$

Furthermore, from the first equation of motion in (2.7) and (5.11) it follows that

$$h\widetilde{d}_k = \frac{\exp(\widetilde{x}_k - x_k) - 1}{1 + g^2 \exp(x_{k+1} - \widetilde{x}_k)}. \tag{5.13}$$

Now equations (5.7), (5.8) follow from (5.12), (5.13) under the parametrization $d_k = \exp(p_k)$. □

The Lax representation for the system (5.7), (5.8) is given by proposition 2 with the following expressions through x_k, $\widetilde{x}_k$: (5.12), (5.13) for d_k, $\widetilde{d}_k$, $c_k = g^2 \exp(x_{k+1} - x_k) d_k$, and

$$\mathfrak{a}_k = \exp(\widetilde{x}_k - x_k)\frac{\big(1 + g^2 \exp(x_{k+1} - \widetilde{x}_k)\big)}{\big(1 + g^2 \exp(x_{k+1} - x_k)\big)}\ \frac{\big(1 + g^2 \exp(x_k - x_{k-1})\big)}{\big(1 + g^2 \exp(x_k - \widetilde{x}_{k-1})\big)} \tag{5.14}$$

$$\mathfrak{b}_k = \exp(\widetilde{x}_k - x_k). \tag{5.15}$$

Indeed, equation (5.14) follows from (5.9), (5.12), and equation (5.15) follows from (3.12), (5.14) and (5.11).

Identifying equations (5.7) and (5.8) with (4.15), (4.16), we obtain a Lagrange function

$$\Lambda_+^{(2)}(\widetilde{x}, x) = \sum_{k=1}^{N} \Phi(\widetilde{x}_k - x_k) + \sum_{k=1}^{N} \big[\Psi(x_{k+1} - \widetilde{x}_k) - \Psi(x_{k+1} - x_k)\big] \tag{5.16}$$

where the two functions $\Phi(\xi)$, $\Psi(\xi)$ are defined by

$$\Phi(\xi) = \int_0^{\xi} \log\left|\frac{\exp(\eta) - 1}{h}\right| \,\mathrm{d}\eta \qquad \Psi(\xi) = \int_0^{\xi} \log\big(1 + g^2 \exp(\eta)\big)\,\mathrm{d}\eta. \tag{5.17}$$

It is easy to see that this Lagrangian function serves as a finite difference approximation to (5.3).

Theorem 4. In the parametrization (5.1) the map (2.10) takes the form of the following two equations:

$$\exp(p_k) = \frac{h\big(1 + g^2\exp(x_k - \widetilde{x}_{k-1})\big)}{\big(1 - \exp(\widetilde{x}_k - x_k)\big)} \tag{5.18}$$

$$\exp(\widetilde{p}_k) = \frac{h\big(1 + g^2\exp(\widetilde{x}_k - \widetilde{x}_{k-1})\big)}{\big(1 - \exp(\widetilde{x}_k - x_k)\big)}\,\frac{\big(1 + g^2\exp(x_{k+1} - \widetilde{x}_k)\big)}{\big(1 + g^2\exp(\widetilde{x}_{k+1} - \widetilde{x}_k)\big)}. \tag{5.19}$$

This implies the same Newtonian equations of motion (1.6) as for the map (2.7).

Proof. From equations (2.10), (2.11) we deduce that

$$\frac{\widetilde{c}_k}{\widetilde{d}_k} = \frac{c_{k+1}}{d_{k+1}}\,\frac{c_k + h\mathfrak{d}_k}{d_k - h\mathfrak{d}_{k-1}}\,\frac{d_{k+1} - h\mathfrak{d}_k}{c_{k+1} + h\mathfrak{d}_{k+1}} = \frac{c_{k+1}}{d_{k+1}}\,\frac{\mathfrak{d}_k}{\mathfrak{d}_{k+1}}.$$

Because $c_k/d_k = g^2\exp(x_{k+1} - x_k)$, this implies that the following quantity does not depend on k:

$$\mathfrak{d}_k\exp(\widetilde{x}_k - x_{k+1}) = \text{constant}.$$

Setting this constant equal to g^2, we obtain

$$\mathfrak{d}_k = g^2\exp(x_{k+1} - \widetilde{x}_k). \tag{5.20}$$

Substituting this formula in the recurrence (2.11), we obtain

$$d_k = \frac{h\big(1 + g^2\exp(x_k - \widetilde{x}_{k-1})\big)}{\big(1 - \exp(\widetilde{x}_k - x_k)\big)}. \tag{5.21}$$

Equations (5.18), (5.21) also imply that

$$d_k - h\mathfrak{d}_{k-1} = \frac{h\big(1 + g^2\exp(\widetilde{x}_k - \widetilde{x}_{k-1})\big)}{\big(1 - \exp(\widetilde{x}_k - x_k)\big)}. \tag{5.22}$$

This last formula, together with the first equation in (2.10), implies that

$$\widetilde{d}_k = \frac{h\big(1 + g^2\exp(\widetilde{x}_k - \widetilde{x}_{k-1})\big)}{\big(1 - \exp(\widetilde{x}_k - x_k)\big)}\,\frac{\big(1 + g^2\exp(x_{k+1} - \widetilde{x}_k)\big)}{\big(1 + g^2\exp(\widetilde{x}_{k+1} - \widetilde{x}_k)\big)}. \tag{5.23}$$

Finally, equations (5.18), (5.19) are equivalent to (5.21), (5.23), due to the parametrization $d_k = \exp(p_k)$. □

The Lax representation for the system (5.18), (5.19) is given by proposition 4 with the following expressions through x_k, $\widetilde{x}_k$: (5.21), (5.23) for d_k, $\widetilde{d}_k$, $c_k = g^2\exp(x_{k+1} - x_k)d_k$, (5.20) for $\mathfrak{d}_k$, and

$$\mathfrak{c}_k = g^2\exp(x_{k+1} - \widetilde{x}_k)\frac{\big(1 - \exp(\widetilde{x}_{k+1} - x_{k+1})\big)}{\big(1 - \exp(\widetilde{x}_k - x_k)\big)}\,\frac{\big(1 + g^2\exp(\widetilde{x}_k - \widetilde{x}_{k-1})\big)}{\big(1 + g^2\exp(\widetilde{x}_{k+1} - \widetilde{x}_k)\big)}.$$

The last formula follows from (3.22), (5.20) and (5.22).

Identifying equations (5.18) and (5.19) with (4.15), (4.16), we obtain a Lagrange function

$$\Lambda_-^{(2)}(\widetilde{x}, x) = -\sum_{k=1}^{N}\Phi(\widetilde{x}_k - x_k) + \sum_{k=1}^{N}\big[\Psi(\widetilde{x}_k - \widetilde{x}_{k-1}) - \Psi(x_k - \widetilde{x}_{k-1})\big] \tag{5.24}$$

with the same functions $\Phi(\xi)$, $\Psi(\xi)$ (equation (5.17)) as before. It is easy to see that this Lagrangian function is a finite difference approximation to (5.5).

6. Parametrization of the mixed brackets by canonically conjugate variables

It turns out that there exist still another parametrization of the variables (c, d) by canonically conjugate variables (x, p) leading to interesting discretizations. As we shall see, these parametrizations lead to Poisson brackets which are linear combinations of the two homogeneous ones (2.3) and (2.5). In some sense (which will be clear from the proofs of the theorems below) these parametrizations are specially designed to obtain nice Newtonian equations from the maps (2.7), (2.10).

We start with the parametrization leading to the linear combination $\{\cdot,\cdot\}_1 + h\{\cdot,\cdot\}_2$, which turns out to admit a nice discrete Newtonian formulation when applied to (2.7). Clearly, we obtain an alternative discretization of the lattice (1.1) in this way. Consider the following parametrization:

$$hd_k = \exp(hp_k) - 1 - h\exp(x_k - x_{k-1}) \qquad c_k = \exp(x_{k+1} - x_k + hp_k). \tag{6.1}$$

(Obviously, in the limit $h \to 0$ we recover the parametrization of the linear bracket (4.2)). Simple calculations show that the Poisson brackets between (c, d) induced by (6.1) read:

$$\{c_{k+1}, c_k\} = hc_{k+1}c_k \qquad \{d_{k+1}, d_k\} = -c_k$$

$$\{d_{k+1}, c_k\} = c_k + hd_{k+1}c_k \qquad \{d_k, c_k\} = -c_k - hd_kc_k$$

which is exactly $\{\cdot,\cdot\}_1 + h\{\cdot,\cdot\}_2$. Let us look at the equations of motion generated by these h-dependent Poisson brackets.

Theorem 5. In the parametrization (6.1) the map (2.7) takes the form of the following two equations:

$$\exp(hp_k) = \exp(\widetilde{x}_k - x_k)\,\frac{\big(1 + h\exp(x_k - x_{k-1})\big)\big(1 - h\exp(x_k - \widetilde{x}_{k-1})\big)}{\big(1 + h\exp(x_{k+1} - x_k)\big)} \tag{6.2}$$

$$\exp(h\widetilde{p}_k) = \exp(\widetilde{x}_k - x_k)\big(1 - h\exp(x_{k+1} - \widetilde{x}_k)\big). \tag{6.3}$$

This implies the Newtonian equations of motion (1.7).

Proof. The crucial observation lies in the following: from relations (6.1) we can extract the following consequence:

$$\exp(hp_k) = 1 + hd_k + \frac{hc_{k-1}}{\exp(hp_{k-1})}.$$

Comparing this with (2.8), we see that $\mathfrak{a}_k$ and $\exp(hp_k)$ satisfy one and the same recurrence relation. Due to the uniqueness of its solution, they must coincide, so that we obtain

$$\mathfrak{a}_k = \exp(hp_k). \tag{6.4}$$

As a consequence, we immediately obtain

$$\mathfrak{a}_k - hd_k = 1 + h\exp(x_k - x_{k-1}) \qquad \mathfrak{a}_k + hc_k = \exp(hp_k)\big(1 + h\exp(x_{k+1} - x_k)\big). \tag{6.5}$$

These expressions, together with (6.1), when substituted in (2.7), allow us to rewrite the latter in the form

$$\begin{aligned} \exp(h\widetilde{p}_k) - h\exp(\widetilde{x}_k - \widetilde{x}_{k-1}) &= \exp(hp_k)\,\frac{1 + h\exp(x_{k+1} - x_k)}{1 + h\exp(x_k - x_{k-1})} - h\exp(x_{k+1} - x_k) \\ \exp(\widetilde{x}_{k+1} - \widetilde{x}_k + h\widetilde{p}_k) &= \exp(x_{k+1} - x_k + hp_{k+1})\,\frac{1 + h\exp(x_{k+2} - x_{k+1})}{1 + h\exp(x_{k+1} - x_k)}. \end{aligned} \tag{6.6}$$

The formula arising when $\exp(h\widetilde{p}_k)$ is excluded from these two equations can, after some manipulation, be written as

$$\exp(x_{k+1}-\widetilde{x}_{k+1}+hp_{k+1})\,\frac{1+h\exp(x_{k+2}-x_{k+1})}{1+h\exp(x_{k+1}-x_k)}+h\exp(x_{k+1}-\widetilde{x}_k)$$

$$=\exp(x_k-\widetilde{x}_k+hp_k)\,\frac{1+h\exp(x_{k+1}-x_k)}{1+h\exp(x_k-x_{k-1})}+h\exp(x_k-\widetilde{x}_{k-1}).$$

So, the expression on the right-hand side is constant, i.e. it does not depend on k. Setting this constant equal to 1, we arrive at equation (6.2). Substituting equation (6.2) in (6.6), we obtain equation (6.3). □

It is not difficult to extract from this proof the expressions for the coefficients of the matrices forming the Lax representation of the system (1.7) following on from proposition 2. Also a Lagrangian formulation of this system can be obtained in a standard way: the Lagrange function corresponding to (6.2), (6.3) is

$$\Lambda_+^{(\mathrm{mixed})}(\widetilde{x},x)=\sum_{k=1}^{N}\frac{(\widetilde{x}_k-x_k)^2}{2h}-h^{-1}\sum_{k=1}^{N}\big[\phi_1(x_{k+1}-\widetilde{x}_k)+\phi_2(x_{k+1}-x_k)\big] \tag{6.7}$$

where

$$\phi_1(\xi)=\int_0^{\xi}\log\big(1-h\exp(\eta)\big)\,\mathrm{d}\eta \qquad \phi_2(\xi)=\int_0^{\xi}\log\big(1+h\exp(\eta)\big)\,\mathrm{d}\eta.$$

This is a finite difference approximation to (4.5) that differs from (4.17).

We now turn to another parametrization that leads to a mixed Poisson bracket, namely to the bracket $\{\cdot,\cdot\}_2-h\{\cdot,\cdot\}_1$. The corresponding formulae are

$$d_k=\exp(p_k)+h\big(1+g^2\exp(x_k-x_{k-1})\big) \qquad c_k=g^2\exp(x_{k+1}-x_k+p_k). \tag{6.8}$$

As can easily be calculated, the resulting Poisson brackets between the variables (c,d) are:

$$\{c_{k+1},c_k\}=c_{k+1}c_k \qquad \{d_{k+1},d_k\}=-hc_k$$

$$\{d_{k+1},c_k\}=d_{k+1}c_k-hc_k \qquad \{d_k,c_k\}=-d_kc_k+hc_k$$

i.e. the linear combination $\{\cdot,\cdot\}_2-h\{\cdot,\cdot\}_1$. The equations arising from (2.10) under this parametrization, naturally approximate the relativistic Toda lattice (1.3).

Theorem 6. In the parametrization (6.8) the map (2.10) takes the form of the following two equations:

$$\exp(p_k)=\frac{h\big(1+g^2\exp(x_k-\widetilde{x}_{k-1})\big)}{\big(\exp(-\widetilde{x}_k+x_k)-1\big)} \tag{6.9}$$

$$\exp(\widetilde{p}_k)=\frac{h\big(1+g^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1})\big)}{\big(\exp(-\widetilde{x}_k+x_k)-1\big)}\,\frac{\big(1+g^2\exp(x_{k+1}-\widetilde{x}_k)\big)}{\big(1+g^2\exp(\widetilde{x}_{k+1}-\widetilde{x}_k)\big)}. \tag{6.10}$$

This implies the Newtonian equations of motion (1.8).

Proof. This time the crucial observation lies in the following: as a consequence of relations (6.8) we have:

$$\frac{c_k}{g^2\exp(x_{k+1}-x_k)}=d_k-h-hg^2\exp(x_k-x_{k-1}).$$

Comparing this with (2.11), we see that $\mathfrak{d}_k$ and $g^2\exp(x_{k+1}-x_k)$ satisfy one and the same recurrence relation. Due to the uniqueness of its solution, they must coincide, so that we obtain

$$\mathfrak{d}_k = g^2\exp(x_{k+1}-x_k). \tag{6.11}$$

As a consequence, we immediately obtain

$$d_k - h\mathfrak{d}_{k-1} = \exp(p_k)+h \qquad c_k + h\mathfrak{d}_k = g^2\exp(x_{k+1}-x_k)\big(\exp(p_k)+h\big). \tag{6.12}$$

Substituting these expressions, together with (6.8), in (2.10), we can express the latter in the form

$$\begin{aligned}
&\exp(\widetilde{p}_k) + hg^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1}) = \exp(p_k)+hg^2\exp(x_{k+1}-x_k)\ \frac{\exp(p_k)+h}{\exp(p_{k+1})+h}\\
&\exp(\widetilde{x}_{k+1}-\widetilde{x}_k+\widetilde{p}_k) = \exp(x_{k+1}-x_k+p_{k+1})\ \frac{\exp(p_k)+h}{\exp(p_{k+1})+h}.
\end{aligned} \tag{6.13}$$

Excluding $\exp(\widetilde{p}_k)$ from these two equations, and after some manipulation, we obtain the formula that can be written as

$$\begin{aligned}
&\frac{\exp(x_{k+1}-\widetilde{x}_{k+1}+p_{k+1})-hg^2\exp(x_{k+1}-\widetilde{x}_k)}{\exp(p_{k+1})+h}\\
&\qquad = \frac{\exp(x_k-\widetilde{x}_k+p_k)-hg^2\exp(x_k-\widetilde{x}_{k-1})}{\exp(p_k)+h}.
\end{aligned}$$

So, the expression on the right-hand side is constant, i.e. it does not depend on k. Setting this constant equal to 1, we arrive at (6.9), which is also equivalent to

$$\exp(p_k)+h = \frac{h\exp(-\widetilde{x}_k+x_k)\big(1+g^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1})\big)}{\exp(-\widetilde{x}_k+x_k)-1}. \tag{6.14}$$

Finally, substituting (6.14) in (6.13), we obtain

$$\exp(\widetilde{p}_k) = \exp(p_{k+1})\ \frac{\big(\exp(-\widetilde{x}_{k+1}+x_{k+1})-1\big)}{\big(\exp(-\widetilde{x}_k+x_k)-1\big)}\ \frac{\big(1+g^2\exp(\widetilde{x}_k-\widetilde{x}_{k-1})\big)}{\big(1+g^2\exp(\widetilde{x}_{k+1}-\widetilde{x}_k)\big)}$$

which, together with (6.9), implies (6.10). □

The Lax representation for (1.8) is given by proposition 4; it is not difficult to extract from the proof above the expressions for the entries of the matrices forming the Lax representation. Also, the Lagrangian formulation of (1.8) easily follows from (6.9), (6.10). The corresponding Lagrange function is

$$\Lambda_-^{(\mathrm{mixed})}(\widetilde{x},x) = \sum_{k=1}^N \Phi(-\widetilde{x}_k+x_k) + \sum_{k=1}^N \big[\Psi(\widetilde{x}_k-\widetilde{x}_{k-1}) - \Psi(x_k-\widetilde{x}_{k-1})\big] \tag{6.15}$$

with the functions $\Phi(\xi)$, $\Psi(\xi)$ given in (5.17).

Let us note that, although equations (1.6), (1.8) are very similar, there exist no obvious changes of variables bringing one of them into another. The only way to do this, i.e. to connect the two corresponding sets of $(x_k,\widetilde{x}_k)$, is to identify the corresponding (c_k,d_k), given for (1.6) by (5.1), (5.18), (5.19), and for (1.8) by (6.8), (6.9), (6.10). The resulting change of variables is a rather non-trivial Bäcklund transformation.

7. Conclusion

The main message of the present paper is as follows. The field of integrable systems of classical mechanics, even in its most extensively studied parts, is far from being exhausted. Namely, the well known flows of the relativistic Toda hierarchy (2.1), (2.2) have a much richer dynamical content than is usually assumed. This is even truer for the recently derived discretizations (2.7), (2.10) of these flows. Namely, different parametrizations of the variables (c, d) by canonically conjugate variables (x, p) (corresponding to the bi-Hamiltonian structure of the relativistic Toda hierarchy) allowed us to derive two new integrable continuous time lattices and four new integrable discretizations, in addition to the previously known ones. Let us summarize the relations between the systems considered in the present paper:

- The lattices (1.1) and (1.3) arise from the flow (2.1) under two different parametrizations of (c_k, d_k) by canonically conjugate variables (x_k, p_k). Hence these two lattices are connected by means of a highly non-trivial Bäcklund transformation. This transformations connects two sets of variables $(x_k, \dot{x}_k)$ and arises when identifying the variables (c_k, d_k) in equations (4.6) and (5.4).
- Analogously, the lattices (1.2) and (1.3) arise both from one and the same flow (2.2). The Bäcklund transformation connecting the variables $(x_k, \dot{x}_k)$ of these two lattices arises when identifying the variables (c_k, d_k) in equations (4.20) and (5.6).
- The discrete flow (2.7) gives rise to the following three discrete integrable systems: (1.4), (1.6), (1.7). They all are related by the Bäcklund transformations which connect the different variables $(x_k, \widetilde{x}_k)$ introduced by equations (4.2), (4.7) for the system (1.4); by equations (5.1), (5.7) for the system (1.6); and by equations (6.1), (6.2) for the system (1.7).
- Finally, the discrete flow (2.10) also gives rise to three discrete integrable systems: (1.5), (1.6), (1.8). The Bäcklund transformations which connect the corresponding sets of variables $(x_k, \widetilde{x}_k)$ are obtained when identifying the variables (c_k, d_k) given by expressions (4.2), (4.21) for the system (1.5); by expressions (5.1), (5.18) for the system (1.6); and by expressions (6.8), (6.9) for the system (1.8).

The methods of this paper can also be used in the simpler situation of the usual Toda lattice, where they also lead to interesting findings. These will be reported in a separate paper.

References

[1] Suris Yu B 1996 A discrete-time relativistic Toda lattice *J. Phys. A: Math. Gen.* **29** 451-65 and references therein